Bound entanglement can be activated

Paweł Horodecki*
Faculty of Applied Physics and Mathematics
Technical University of Gdańsk, 80–952 Gdańsk, Poland

Michał Horodecki** and Ryszard Horodecki***
Institute of Theoretical Physics and Astrophysics
University of Gdańsk, 80–952 Gdańsk, Poland

Bound entanglement is the noisy entanglement which cannot be distilled to a singlet form. Thus it cannot be used alone for quantum communication purposes. Here we show that, nevertheless, the bound entanglement can be, in a sense, pumped into single pair of free entangled particles. It allows for teleportation via the pair with the fidelity impossible to achieve without support of bound entanglement. The result also suggests that the distillable entanglement may be not additive.

Pacs Numbers: 03.65.Bz

Despite deep research, quantum entanglement still astonishes even specialists, producing highly nonintuitive effects such as quantum paralelism [1], quantum cryprography [2], quantum dense coding [3], quantum teleportation [4], reduction of communication complexity [5]. In practice, one usually deals with noisy entanglement represented by *mixed* states of composite system. The latter are entangled (inseparable) if they are not mixtures of product states [6,7]. However, the mixed state entanglement cannot be used directly for quantum communication purposes. For this reason the first example procedure of distillation of it to useful singlet form represented by the two spin-$\frac{1}{2}$ singlet state $\Psi_- = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ has been provided by Bennet *et. al* [8] and discussed later in [9]. Similar procedure has been applied to quantum privacy amplification [10]. Subsequently, it has been shown [11] that, noisy entanglement of two spin-$\frac{1}{2}$ system, however small, can be distilled to a the singlet form. Then it was naturally supposed that the same is possible for larger systems. However, quite recently, it has been shown that begining with two spin-1 systems, quantum mechanics implies existence of two qualitatively different kinds of noisy entanglement [12]: apart from the "free" entanglement which is distillable there is a "bound" one which by no means can be brought to the singlet form. The curiosity of the bound entangled states is that to produce them one needs some amount of pure entanglement, while any, however little amount of it cannot be recovered back from them. The bound entanglement is closely connected with Peres separability test [13] (see also [14]). In particular, it has been shown [12] that if an inseparable state satisfies Peres criterion, then its entanglement is bound.

The existence of bound entanglement involves new questions concerning local realism and quantum information. However there is a question closely related to the practical topics. Namely one can simply ask: Can the bound entanglement be somehow activated to produce *any* effect useful in quantum communication? In this paper we show that the bound entanglement can be, in a sense, liberated, giving, in particular, some chance of improving the transmission of quantum information and suggesting existence of qualititatively new processes in mixed entanglement domain.

Before we state the main results of this paper let us recall that quite recently [16,17] it has been pointed out that mixed free entanglement may have some disavantage as it cannot be distilled *noncollectively* i.e. by acting over any given pair of particles separately. It particular it means that in some cases given *single* pair of two spins particles in free entangled (FE) state ϱ_{in}, using only quantum local operations (QL) and classical communication (CC), cannot make the *fidelity* of the resulting state ϱ_{out}

$$F(\varrho) = \langle\Psi_+|\varrho|\Psi_+\rangle, \quad |\Psi_+\rangle = \frac{1}{\sqrt{2s+1}}\sum_{i=0}^{2s}|i\rangle|i\rangle \qquad (1)$$

arbitrary close to 1. This is an important point as quantity (1), measuring how close is ϱ to maximally entangled two spin-s state Ψ_+, plays central role in the teleportation scheme [4] if applied to mixed states [15].

Now, let us explain the main result of this paper. We consider just a *single* pair of spin-1 particles in a mixed state ϱ shared by paradigmatic, spatially separated Alice and Bob who are allowed to make any QLCC operations. The state ϱ is taken to be free entangled (FE), but its entanglement is chosen to be so weak that in the case of single pair no QLCC operations can increase its fidelity upon some bound $C < 1$. We then introduce some new bound entangled (BE) states and show that if, in addition, Alice and Bob are provided with a large supply of pairs in those states, then they can skip the border C making now the fidelity of original FE pair arbitrary close to 1 with nonzero probability. We shall hereafter call the process of making the fidelity F arbitrary close to unity *quasi-distillation*, as, in contrast with the original distillation idea, we allow the number of initial pairs

and the probability of success to depend on the required final F. The key point of the presented result is here that the distinguished FE pair as well as the set of all BE pairs cannot be quasi-distilled themselves. However, putting them together produces new quality from which the state with arbitrary good fidelity already can be obtained. The revealed process can be viewed as a kind of entanglement transfer from BE pairs into FE pair. After the presentation of details of the effect (see below) we adress the question of possible relevance of the effect for quantum communication and show that some transfer which is impossible with FE pair alone sometimes can be done with aid of bound entanglement supply. This supports the hope for some usefulness of bound entanglement for quantum communication purposes. Finally we discuss possible relevance of the effect in the context of the original distillation idea. In particular we conclude that, in the present context, it can not be excluded that the distillable entanglement [9] is not additive.

To ilustrate details of scheme consider the case of two spin-1 particles. The state of any particle can be described using the three-dimensional Hilbert space spanned by the basis states $|0\rangle, |1\rangle, |2\rangle$ corresponding to antiparallel, perpendicular and parallel orientation of particle spin with respect ot the z axis. This means, in particular, that we put s=1 in formula (1).

For our purposes let us introduce mixed separable states:

$$\sigma_+ = \frac{1}{3}(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|)$$
$$\sigma_- = \frac{1}{3}(|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|) \qquad (2)$$

Suppose now that Alice nad Bob share *single* pair of spin-1 particles in the free entangled mixed state

$$\varrho_{free} = \varrho(F) \equiv F|\Psi_+\rangle\langle\Psi_+| + (1-F)\sigma_+, \quad 0 < F < 1 \qquad (3)$$

In fact, it it is easy to see that the state is free entangled. Namely after action of the local projections $(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$ we get inseparable 2×2 state. Such a result is known [12] to be sufficient for distillability of entanglement from the state, hence for the fact that $\varrho(F)$ contains free entanglement. On the other hand it can be shown [18] that the state can never be quasi-distilled noncollectively i.e. no QLCC performed on one pair in $\varrho(F)$ can increase its fidelity upon some C (we do not present the proof here, as it is a rather technical task and requires some new approach then the ones applied so far).

Suppose, however, that apart from the state $\varrho(F)$ Alice and Bob have a large number of pairs of particles in the following state [19]:

$$\sigma_\alpha = \frac{2}{7}|\Psi_+\rangle\langle\Psi_+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_- \qquad (4)$$

Those states admit simple characterization with respect to the parameter $2 \leq \alpha \leq 5$:

$$\sigma_\alpha \ \text{ is } \left\{ \begin{array}{c} \text{separable for } \ 2 \leq \alpha \leq 3 \\ \text{bound entangled for } \ 3 < \alpha \leq 4 \\ \text{free entangled for } \ 4 < \alpha \leq 5 \end{array} \right. \qquad (5)$$

Let us briefly justify the above characterisation. It is easy to point out separability of σ_α states for first region of parameter $2 \leq \alpha \leq 3$. In fact then it can be written as a mixture of separable states (recall that separable states form the convex set) $\sigma_\alpha = \frac{6}{7}\varrho_1 + \frac{\alpha-2}{7}\sigma_+ + \frac{3-\alpha}{7}\sigma_-$. Here $\varrho_1 = (|\Psi_+\rangle\langle\Psi_+| + \sigma_+ + \sigma_-)/3$, which has been explicitly represented as mixture of product states in Ref. [7]. It is even more easy to find the free entanglement of σ_α in the last region $4 < \alpha \leq 5$, as it can be done in the same way as for the states (3). For intermediate region $3 < \alpha \leq 4$ direct calculation shows that σ_α satisfies Peres separability criterion of positive partial transposition [13]. Nevertheless in this case the state is inseparable, and then, as shown in Ref. [12], it is bound entangled. Here, instead of direct proving of this inseparability, we will rather show that such a states can produce the effect which cannot come from any separable state. Namely we shall show that if only the states σ_α Alice and Bob share have $3 < \alpha \leq 4$ then they can quasi-distill the state ϱ_{free}. Note, that it would *not* be possible if the state were separable. Indeed, any usage of separable state together with QLCC action could be interpreted as some new QLCC action *alone*, since the separable state itself can be produced by means of some QLCC operation. However, as it was mentioned before, *no* QLCC on single pair in state (3) can quasi-distill it. Thus, the possibility of quasi-distillation of a single pair $\varrho(F)$ with help of state σ_α, $3 < \alpha \leq 4$ will be at the same time the proof that the latter is bound entangled. Note that any initial supply of BE states, if represents the only entanglement in the process, cannot be quasi-distilled [20].

Consider now the protocol of quasi-distillation. Recall that Alice and Bob share one pair in the FE state (3) and large supply of pairs in BE states (4). They can procees repeating the folowing two step procedure which is, in practice, the direct 3×3 analogue of the one used in destillation of entanglement [8,11,21]:

(i) They take the free entangled pair in the state $\varrho_{free}(F)$ and one of the pairs being in the state σ_α. They perform the bilateral XOR operation $U_{BXOR} = U_{XOR} \otimes U_{XOR}$, each of them treating the member of free (bound) entangled pair as a source (target). Recall here that the unitary XOR gate introduced in [8], and used in generalised form in [21,22] is defined as

$$U_{XOR}|a\rangle|b\rangle = |a\rangle|b \oplus a\rangle, \quad b \oplus a = (b+a)\mathrm{mod}N \qquad (6)$$

where initial state $|a\rangle$ ($|b\rangle$) corresponds to source (target) state.

(ii) After that Alice and Bob measure the in their laboratories the z-axis spin components of the members of source pair. Then they compare their results via phone. If the results are the same they discard only target pair, coming back with the, as we shall see, improved source pair to the first step (i). If the compared results appear to be different they have to discard both pairs and then the trial of improvement of F fails.

By virtue of high symetry of the states (4), (3) it is easy to see that, conditioned that Alice and Bob get the same results of their measurement in step (ii), the above protocol leads with nonzero probability

$$P_{F\to F'} = \frac{2F + (1-F)(5-\alpha)}{7}, \tag{7}$$

to the tranformation $\varrho(F) \to \varrho(F')$ where the improved fidelty F' amounts to

$$F'(F) = \frac{2F}{2F + (1-F)(5-\alpha)} \tag{8}$$

If only α is greater than 3 the above continuous function of F exeeds the value of F on the whole region $(0,1)$. Thus the succesfull repeating of the steps (i-ii) produces the sequence of source fidelities $F_n \to 1$. The probability of achieving any fidelity F_n is $P_n = (P_{F\to F'})^n$ hence it is *nonzero* for any n. Thus all the states (4) with $3 < \alpha \le 5$ allow us to quasi-distill state (3). In particular the effect holds for region $3 < \alpha \le 4$ confirming that the target state (4) is inseparable, hence bound entangled in this region. On the contrary for the region $2 \le \alpha \le 3$ the iteration of the formula (8) decreases fidelity. This dramatic qualitatively change reflects the fact that then Alice's and Bob's large supply of pairs is in separable states which, as it was indicated before, cannot help to quasi-distill pair in state (3). It is remarkable result as it shows that seemingly usless bound entanglement can be, in a sense, pumped into single pair of free entangled particles. We expect similar effect for other bound entangled states like those introduced in Ref. [7].

Let us discuss the physical meaning of the result. First we shall point out an interesting connection of the result with the special kind of quantum communication which is teleportation. Recall that any quantum state of composite system ϱ can be regarded as a channel in the process of the teleportation. The idea is that Alice posses one particle in unknown state ψ and one member of pair being in the state ϱ. Bob posses another member of the pair. After performing some deliberately chosen QLCC operation Bob finds his particle in the state resambling, at least to some degree, the initial unknow state ψ of Alice particle. The fidelity of transmission of the state is measured by the transfer fidelity $f = \overline{\langle\psi|\Lambda_\varrho(\psi)|\psi\rangle}$ where $\Lambda_\varrho(\psi)$ represents Bob's particle state after the whole procedure [15] and the bar stands for average over all possible input states ψ. If the state ϱ which forms the quantum teleportation channel is the maximally entangled state, then optimally chosen QLCC guarantees $\Lambda_\varrho(\psi) = |\psi\rangle\langle\psi|$ and the transfer fidelity f is equal to unity. However in general, f can be lesser than 1. One can prove two simple connections between quantity F and fidelity f of teleportation transfer. Namely [18]: (i$'$) if F of a state ϱ_{out} obtained form ϱ by any QLCC operation is bounded by $C(\varrho) < 1$ than f of any teleporting procedure through the new state ϱ_{out} is certainly bounded by some other constant $d(\varrho) < 1$; (ii$'$) if for some family of states $\varrho(F_n)$ the fidelity F_n converges to 1 then under original teleportation procedure [4] the teleportation fidelity f_n through $\varrho(F_n)$ also converges to 1.

Let us now consider our result in the context of these two facts. For given $\varrho(F)$ we know that no QLCC can increase F over some threshold $C(\varrho(F)) < 1$. Hence according to (ii$'$) the teleportation transfer fidelity f after any QLCC operation is also bounded by some $d(\varrho) < 1$. Suppose now that Alice wants to teleport unknown spin-1 state $|\phi\rangle$ to Bob but only if she *is sure* that the state is teleported with fidelity f better than some fixed bound f_r satisfying $d(\varrho) < f_r < 1$. If the FE state (3) is *the only* entangled state shared by her and Bob then she will *never* decide to teleport, as her requirement cannot be satisfied. However, according to the results of iteration of scheme (i-ii) and the item (i$'$), if apart from ϱ_{free} Alice and Bob share a lot of BE states (4) then still there is some nonzero chance that, after some LQCC operations Alice can teleport, being *sure* that her transmission has the required fidelity $f > f_r$. Thus we see that bound entanglement can lead to qualitative improvement of the processes of quantum communication.

From the formula (8) it follows that the bound entanglement contained in the target state ($3 < \alpha \le 4$) behaves qualitatively *in the same way* as a free entanglement ($3 < \alpha \le 4$). Moreover the effect would not hold if only the single source pair were in bound entangled state. But even if it is free entangled but alone, quasi-distillation will not succeed. In contrast, the interaction between with large number of BE states allows to make its fidelity arbitrary close to 1.

A way of interpreting the results presented above is suggested by entanglement-energy analogy [12]. Namely the situation is somewhat similar to the processes which need an initial supply of some amount of energy to be run. Here the role of the extra initial energy is played by the single free entangled pair, which is allows to run the process of drawing entanglement from the BE pairs.

In Ref. [12] the analogy entanglement-energy was stated quantitatively, where the analogue of useful (free) energy was the distillable entanglement D. Recall that the distillable entanglement $D(\varrho)$ denotes the maximal number of singlet pairs per input pair which can be produced by means of QLCC operations from large number of pairs in the state ϱ. Now we can expect some other

effect being in more strict analogy with energy exchange processes. Namely, we expect that distillable entanglement [9] $D(\varrho)$ may be *non-additive*. In fact, by definition [12], for any FE state ϱ_{free} one has $D(\varrho_{free}) > 0$ while for BE ones $D(\varrho_{bound}) = 0$. Note that the presented quasi-distillation scheme involves some kind of entanglement transfer from BE pairs into the FE one. It suggests that we may have $D(\varrho_{free} \otimes \varrho_{bound}) > D(\varrho_{free})$. But the latter is simply the sum $D(\varrho_{free}) + D(\varrho_{bound})$ as the last term vanish by definition. This would really mimic a strange algebra in which $0+1$ would be greater than 1. Then the bound entanglement which is *not distillable at all if alone* could be distillable *through* free entanglement: the latter would be the window allowing to liberate the former. In terms of the mentioned analogy, the bound entanglement would perform for us useful informational work, if supported by, perhaps small supply free entanglement. Then, the role of the latter would be to activate the bound one.

Even more probable effect strongly suggested by the present results is the following. Suppose that we enrich the actions Alice and Bob are conventionally allowed to do. Namely, apart from performing local quantum operations and classical communication, we allow them to share publicly any amount of bound entangled pairs. Now, what have shown in this paper is that the new class of operations (call it LQCC+BE) is significantly more powerful than the LQCC operations alone. Now one expects that the distillable entanglement within this new paradigm can be strictly greater than the conventional one i.e. we would have $D_{LQCC+BE} > D_{LQCC}$.

Finally, note that our discussion benefit from two opposite points of view. In one of them we treat the bound entanglement as some supplement which helps to handle with the free one, while in the other one, the basis is bound entanglment, while the free one is only to activate it. We believe that both perspectives will be useful for further inverstigation of the role of bound entanglement in quantum information theory.

M. H. and P. H. gratefully acknowledge the support from Foundation for Polish Science.

---

* E-mail address:pawel@mifgate.mif.pg.gda.pl
** E-mail address: michalh@iftia.univ.gda.pl
*** E-mail address: fizrh@univ.gda.pl

[1] D. Deutsch, Proc. R. Soc. London A **425**, 73 (1989).
[2] A. Ekert, Phys. Rev. Lett. 67 (1991) 661.
[3] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69 (1992) 2881.
[4] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993), for experimental realisation see D. Bouwmeester, J.-W. Pan, K. Mattle, M. Elbl, H. Weinfurter and A. Zeilinger, Nature (London) **390**, 575 (1997); D. Boschi, S. Brance, F. De Martini, L. Hardy and S. Popescu, Phys. Rev. Lett. **80**, 1121 (1998).
[5] R. Cleve and H. Burhmann, Phys. Rev. A **56**, 1201 (1997); L. K. Grover, Report No. quant-ph/9704012.
[6] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[7] P. Horodecki, Phys. Lett. A, **232**, 333 (1997).
[8] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[9] C. H. Bennett, D. P. DiVincenzo, J. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3814 (1997).
[10] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett. **77**, 2818 (1996).
[11] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
[12] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[13] A. Peres, Phys. Rev. Lett. **76**, 1413 (1996);
[14] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **1** 223 (1996) .
[15] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).
[16] N. Linden, S. Massar and S. Popescu, Report No. quant-ph/9805001.
[17] A. Kent, Report No. quant-ph/9805088.
[18] P. Horodecki, M. Horodecki, R. Horodecki, "Quasi-distillation with single pair operations", in preparation.
[19] The states were constructed on the basis of the E. Størmer results on positive maps [Proc. Amer. Math. Soc. **86**, 402 (1982)]. (The connection between positive maps and the problem of separability of mixed states has been established in Ref. [14])
[20] Indeed, from however large supply of BE pairs one cannot produce a spin-s pair of fidelity exceeding $\frac{1}{2s+1}$ because states with such fidelity are FE (as shown in Ref. [21]).
[21] M. Horodecki and P. Horodecki Report No. quant-ph/9708015.
[22] D. Gottesmann, Report No. quant-ph/98002007.